## LECTURE 5 - HADRON STATES \&

## CONSEQUENCES OF GROUP THEORY

## Contents

- Hadron States in SU(3) flavour
- Hadron States in SU(6) flavour®spin
- Consequences of Group Theory
- The Standard Model
- Beyond the Standard Model


## Messages

Group Theory provides

- A description of the hadron multiplets
- The framework for the Standard Model
- Indications for a unified description of the Standard Model and new physics


## Hadron States in SU(3) flavour [11]

|  | I | $\mathrm{I}_{3}$ | S | B | $\mathrm{Y}=\mathrm{S}+\mathrm{B}$ | $\mathrm{Q}=\mathrm{I}_{3}+\mathrm{Y} / 2$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| u | $1 / 2$ | $+1 / 2$ | 0 | $1 / 3$ | $+1 / 3$ | $+2 / 3$ |
| d | $1 / 2$ | $-1 / 2$ | 0 | $1 / 3$ | $+1 / 3$ | $-1 / 3$ |
| s | 0 | 0 | -1 | $1 / 3$ | $-2 / 3$ | $-1 / 3$ |



$\mathrm{SU}(3)$ symmetry implies nothing can distinguish between different quark states.

## Mesons



SU(3)
$\otimes$

$3 \otimes \overline{3}$
$=$

8

$\begin{array}{ll}\oplus & \square \\ & \square \\ \oplus & 1\end{array}$
Mixed
Totally
Symmetry
Antisymmetric


Octet - c.f. Gluon octet
Singlet - c.f. colourless meson state

The two $\eta$ mesons have the same quantum numbers: $I=0, I_{3}=0, S=0, B=0, Y=0, Q=0$ albeit that they have different forms under $S U(3)_{\text {flavour }}$ - one transforms as an 8 , the other as a 1.

Because the $\operatorname{SU}(3)_{\text {flavour }}$ is not exact, the observed mass eigenstates of the complete Hamiltonian are mixtures of $\eta_{8}$ and $\eta_{1}$ :

$$
\begin{aligned}
& \eta=\cos \theta_{\mathrm{p}} \eta_{8}-\sin \theta_{\mathrm{p}} \eta_{1} \\
& \eta^{\prime}=\sin \theta_{\mathrm{p}} \eta_{8}+\cos \theta_{\mathrm{p}} \eta_{1}
\end{aligned}
$$

Experimentally, mixing angle $\theta_{\mathrm{p}} \approx-20^{\circ}$
$\pi^{0}$ does not mix since it has $\mathrm{I}=1$.


## Meson Spin

The particles identified so far are the pseudoscalar mesons with $J=0$, corresponding to a spin state $\frac{1}{\sqrt{2}}(\uparrow \downarrow-\downarrow \uparrow)$.

There is a set of heavier particles, the so-called vector mesons with $J=1$ :



As with the $\eta$ mesons, the $\omega$ mesons mix:

$$
\begin{aligned}
& \omega=\cos \theta_{v} \omega_{8}-\sin \theta_{v} \omega_{1} \\
& \phi=\sin \theta_{v} \omega_{8}+\cos \theta_{v} \omega_{1}
\end{aligned}
$$

The mixing is such that $\phi$ is almost pure $s \bar{s}$ :

$$
\begin{aligned}
& \omega \approx \frac{1}{\sqrt{2}}(u \bar{u}+d \bar{d}) \\
& \phi \approx s \bar{s}
\end{aligned}
$$

Sometimes, it can be helpful to express the meson wavefunctions in a symmetrised form - the flavour symmetry is described by the G-parity.
E.g.
$\pi^{+}=\frac{1}{\sqrt{2}}(u \bar{d}+\bar{d} u) \cdot \frac{1}{\sqrt{2}}(\uparrow \downarrow-\downarrow \uparrow) \quad G=-1$
$\rho^{+}=\frac{1}{\sqrt{2}}(u \bar{d}-\bar{d} u) \cdot\left\{\begin{array}{c}\uparrow \uparrow \\ \left.\frac{1}{\sqrt{2}} \uparrow \downarrow+\downarrow \uparrow\right) \\ \downarrow \downarrow\end{array}\right\} \quad G=+1$

## Baryons



What are these states?
A decuplet and one octet are seen, but not a singlet.

## Decuplet



Note these are the heavier $\Sigma$ and $\Xi$ states.
By observation, these states have $\mathrm{J}=3 / 2$.

These states are symmetric.

$$
\begin{array}{|l|l|l|}
\hline u & u & u \\
\hline \hline u & u & d
\end{array} \quad \sim \quad \sim \frac{1}{\sqrt{3}} \text { (uuud + udu + duu) }
$$

But the state $\Delta^{++} \sim$ uuu $\cdot \uparrow \uparrow \uparrow$ - which is a symmetric combination of fermions. The fermionic symmetry is restored by adding the antisymmetric colour wave-function:

$$
\frac{1}{\sqrt{6}}(r b g+g r b+b g r-r g b-b r g-g b r)
$$

Quark content $\quad$ Baryon $\quad I_{3} \quad$ S

| $u$ | $u$ | $u$ |
| :--- | :--- | :--- |
| $u$ | $u$ | $d$ |
| $u$ | $d$ | $d$ |
| $u$ | $d$ | $d$ |
| $d$ | $d$ | $d$ |

$\Delta^{++}$
$\Delta^{+}$
$\Delta^{0}$
$\Delta^{-}$
$3 / 2$
$3 / 2$
$3 / 2$
$3 / 2$

| $+3 / 2$ | 0 |
| :--- | :--- |
| $+1 / 2$ | 0 |
| $-1 / 2$ | 0 |
| $-3 / 2$ | 0 |


| $u$ | $u$ | $s$ |
| :--- | :--- | :--- |
| $u$ | $d$ | $s$ |
| $d$ | $d$ | $s$ |
| $d$ | $d$ |  |

$\Sigma^{+}$

| +1 | -1 |
| ---: | ---: |
| 0 | -1 |
| -1 | -1 |


| u | s | s | $\Xi^{0}$ | $1 / 2$ | $+1 / 2$ | -2 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| d | s | s | $\Xi^{-}$ | $1 / 2$ | $-1 / 2$ | -2 |
|  |  |  |  |  |  |  |
| s | s | s | $\Omega^{-}$ | 0 | 0 | -3 |

## Octet



Note these are the lighter $\Sigma$ and $\Xi$ states.
By observation, these states have $J=1 / 2$

How are the octets derived?
$\square$
$\otimes \square \otimes \square=$


| 1 | 2 |
| :--- | :--- |


| $\oplus$ | 1 |
| :--- | :--- |
| 2 |  |

$\otimes$

$\oplus$

Totally Symmetric

$\oplus$

Mixed
Symmetry
Symmetric
in $1 \leftrightarrow 2$
\}


Mixed Symmetry Antisymmetric in $1 \leftrightarrow 2$
$\oplus$


Totally Antisymmetric

Consider a proton ~ uud:

| $\mathrm{u}_{1}$ | $\mathrm{u}_{2}$ | Symmetric in TL $\leftrightarrow T R$; antisymmetric in TL $\leftrightarrow$ BL: |
| :---: | :---: | :---: |
| $\mathrm{d}_{3}$ |  | $\sim\left(u_{1} u_{2}+u_{2} u_{1}\right) \otimes d_{3}=u_{1} u_{2} d_{3}-d_{3} u_{2} u_{1}+u_{2} u_{1} d_{3}-d_{3} u_{1} u_{2}$ <br> ~ uud - duu |
| $\mathrm{u}_{1}$ | $\mathrm{U}_{3}$ | Antisymmetric in TL $\leftrightarrow T R$; symmetric in TL $\leftrightarrow$ BL: |
| $\mathrm{d}_{2}$ |  | $\begin{aligned} & \sim\left(u_{1} d_{2}-d_{2} u_{1}\right) \otimes u_{3}=\left(u_{1} d_{2} u_{3}+u_{3} d_{2} u_{1}\right)-\left(d_{2} u_{1} u_{3}+u_{3} u_{1} d_{2}\right) \\ & \sim 2 u d u-d u u-u u d \end{aligned}$ |

Usually the symmetry is expressed with respect to the first two particles:

$$
\frac{1}{\sqrt{2}}(u d-d u) u \text { or } \frac{1}{\sqrt{6}}\{(u d+d u) u-2 u u d\}
$$

So which is the wavefunction for a proton?

$$
\frac{1}{\sqrt{2}}(u d-d u) u \text { or } \frac{1}{\sqrt{6}}\{(u d+d u) u-2 u u d\}
$$

There appears to be an ambiguity - which will be resolved when we look at $\operatorname{SU}(6)$.
For the octet states with two identical quark flavours, we will find analogous wavefunctions to the above (with the same ambiguity).
However, there will be different combinations for the neutral combinations of ( $\mathrm{u}, \mathrm{d}, \mathrm{s}$ ) corresponding to $\Lambda$ and $\Sigma^{0}$.

Let us count the states of 3 quarks:

```
All same flavour e.g. uuu
3 ways of selecting quarks; 1 permutation of each selection }=>3\mathrm{ states
Members of 10.
```

Two same flavour e.g. uud
6 ways of selecting quarks; 3 permutation of each selection $\Rightarrow 18$ states
Members of $\mathbf{1 0}, \mathbf{8}_{\mathrm{S}}, \mathbf{8}_{\mathrm{A}}$.
All different flavours e.g. uds
1 way of selecting quarks; 6 permutation of each selection $\Rightarrow 6$ states
Members of $\mathbf{1 0}, \mathbf{8}, \mathbf{8}_{\mathrm{A}}, \mathbf{1}$
A total of $27=3 \times 3 \times 3$ states.

| Quark content |  | Baryon | I | $\mathrm{I}_{3}$ | S |
| :---: | :---: | :---: | :---: | :---: | :---: |
| u | $u$ | $p$ | 1/2 | +1/2 | 0 |
| d |  |  |  |  |  |
| u | d | n | 1/2 | -1/2 | 0 |
| d |  |  |  |  |  |
| u | $u$ | $\Sigma^{+}$ | 1 | +1 | -1 |
| s |  |  |  |  |  |
| u | d | $\Sigma^{0}$ | 1 | 0 | -1 |
| s |  |  |  |  |  |
| d | d | $\Sigma^{-}$ | 1 | -1 | -1 |
| s |  |  |  |  |  |
| u | d | $\Lambda$ | 0 | 0 | -1 |
| s |  |  |  |  |  |
| $u$ | S | $\Xi^{0}$ | 1/2 | +1/2 | -2 |
| s |  |  |  |  |  |
| d | S | $\Xi^{-}$ | 1/2 | -1/2 | -2 |
| s |  |  |  |  |  |

## Hadron States in SU(6) flavour $_{\text {spin [15] }}$

In SU(3) flavour, for the baryons, it is not obvious
a. how to assign the octet wavefunctions
b. why there is no flavour singlet

Rather than $\operatorname{SU}(3)_{\text {flavour }} \otimes \operatorname{SU}(2)_{\text {spin }}$, consisting of $\{u, \mathrm{~d}, \mathrm{~s}\} \otimes\{\uparrow, \downarrow\}$, we consider
$\mathrm{SU}(6)_{\text {flavour } \otimes \text { spin }}$, consisting of the states $\{u \uparrow, \mathrm{~d} \uparrow, \mathrm{~s} \uparrow, \mathrm{u} \downarrow, \mathrm{d} \downarrow, \mathrm{s} \downarrow\}$ - all of which are considered indistinguishable.

The flavour-spin states will be combined with an $\mathrm{SU}(3)_{\text {colour }}$ singlet:

$$
\frac{1}{\sqrt{6}}(r b g+g r b+b g r-r g b-b r g-g b r)
$$

This is antisymmetric - so the flavour $\otimes$ spin states will need to be symmetric if they are to describe identical fermions (in particular for the uuu, ddd and sss states).

| $\mathrm{SU}(3)_{\text {flavour }}$ | $3 \otimes 3 \otimes 3=$ | 10 | $\oplus$ | 8 | $\oplus$ | 8 | $\oplus$ | 1 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  |  | $\phi_{\mathrm{S}}$ |  | $\phi_{\mathrm{M}, \mathrm{S}}$ |  | $\phi_{\mathrm{M}, \mathrm{A}}$ |  | $\phi_{\mathrm{A}}$ |
| $\mathrm{SU}(2)_{\text {spin }}$ | $2 \otimes 2 \otimes 2=$ | 4 | $\oplus$ | 2 | $\oplus$ | 2 |  |  |
|  |  | $\chi_{\mathrm{S}}$ |  | $\chi_{\mathrm{M}, \mathrm{S}}$ |  | $\chi_{\mathrm{M}, \mathrm{A}}$ |  |  |

$\phi_{s}$ is symmetric under exchange of all 3 quarks.
$\phi_{\mathrm{M}, \mathrm{s}}$ is symmetric under exchange of only 2 quarks, etc.

We can combine the flavour and spin wavefunctions to create states of new symmetry:

|  |  | $\chi_{\text {spin }}$ |  |
| :---: | :---: | :---: | :---: |
|  |  | S |  |
|  |  |  |  |
| flavour | S |  |  |
|  | M | S | M |
|  | A | M | $\mathrm{S}, \mathrm{M}, \mathrm{A}$ |
|  | A | M |  |

Note: there is no $\chi_{\mathrm{A}}$ to connect to $\phi_{\mathrm{A}}$.
We construct symmetric states:
Decuplet $(\mathrm{J}=3 / 2): \quad \phi_{s} \chi_{\mathrm{s}} \quad 10$ flavour states $\otimes 4$ spin states $=40$ states
Octet $(J=1 / 2): \quad \frac{1}{\sqrt{2}}\left(\phi_{M, S} \chi_{M, S}+\phi_{M, A} \chi_{M, A}\right) \quad 8$ flavour states $\otimes 2$ spin states $=16$ states
A total of 56 states.


## Examples

$\phi_{s} \chi_{s}$

- $\Delta^{++}, J_{z}=3 / 2$
uuu $\uparrow \uparrow \uparrow$
- $\Delta^{+}, J_{z}=1 / 2$
$\frac{1}{\sqrt{3}}($ uud + udu + duu $) \frac{1}{\sqrt{3}}(\uparrow \uparrow \downarrow+\uparrow \downarrow \uparrow+\downarrow \uparrow \uparrow)$
$\frac{1}{\sqrt{2}}\left(\phi_{M, S} \chi_{M, S}+\phi_{M, A} \chi_{M, A}\right)$
- $p, J_{z}=1 / 2$
$\frac{1}{\sqrt{2}} \frac{1}{\sqrt{6}}\{(u d+d u) u-2 u u d\} \frac{1}{\sqrt{6}}\{(\uparrow \downarrow+\downarrow \uparrow) \uparrow-2 \uparrow \uparrow \downarrow\}+\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}}(u d-d u) u \frac{1}{\sqrt{2}}(\uparrow \downarrow-\downarrow \uparrow) \uparrow$
$=\frac{1}{\sqrt{2}}\left\{+\frac{2}{3} u d u-\frac{1}{3} d u u-\frac{1}{3} u u d\right\} \uparrow \downarrow \uparrow+\frac{1}{\sqrt{2}}\left\{-\frac{1}{3} u d u+\frac{2}{3} d u u-\frac{1}{3} u u d\right\} \downarrow \uparrow \uparrow+\frac{1}{\sqrt{2}}\left\{-\frac{1}{3} u d u-\frac{1}{3} d u u+\frac{2}{3} u u d\right\} \uparrow \uparrow \downarrow$
$=\frac{1}{\sqrt{2}}\left\{u \uparrow u \uparrow d \downarrow+u \uparrow d \downarrow u \uparrow+d \downarrow u \uparrow u \uparrow-\frac{1}{3}(u u d+u d u+d u u)(\uparrow \uparrow \downarrow+\uparrow \downarrow \uparrow+\downarrow \uparrow \uparrow)\right\}$
It can be seen that this term is indeed symmetric.


## Final Comments [16.3,16.4]

One can add heavier quarks to $\{\mathrm{u}, \mathrm{d}, \mathrm{s}\}$ :

```
+C SU(4)flavour
+ b SU(5)flavour
+t SU(6)flavour
```

However, since these quarks are much heavier than their lighter brothers and exceed the QCD scale $\Lambda_{Q C D}$, the symmetry is badly broken.

It is best to treat the heavy quarks $Q$ separately from the lighter ones $\{q\}$ :
Mesons: $Q \bar{q}$ where $q$ transforms according to $S U(3)$
Baryons: $\mathrm{Qq}_{1} \mathrm{q}_{2}$ where $\mathrm{q}_{1}$ and $\mathrm{q}_{2}$ transform according to $\mathrm{SU}(3)$

So far, all the hadron states considered have $\mathrm{L}=0$ (s-wave).
Excited states can be obtained by considering higher orbital angular momentum.

## Consequences of Group Theory

$\mathrm{SU}(3)_{\text {colour }}$

- Z Decay

Z $\rightarrow$

$$
\begin{aligned}
& 3 I^{+} l^{-} \\
+ & e, \mu, \tau \\
+ & 5 \times 3 \mathrm{v} \bar{q} \bar{q} \\
& e, \mu, d, c, s, t, b \otimes 3 \text { colours }
\end{aligned}
$$

So the branching ratio for neutrinos is $3 / 21=1 / 7$ not $3 / 11$ (plus ElectroWeak factors).
Significant when measuring visible cross-section at LEP - led to conclusion $\mathrm{N}_{\mathrm{v}}=3$.

- Anomalies - cancellations required for renormalisable theories

$$
\begin{array}{lllr}
\sum_{\text {fermions }} Q=0 & & \\
& & \\
e, \mu, \tau & 3 \times(-1) & -3 \\
v_{e}, v_{\mu}, v_{\tau} & 3 \times(0) & = & 0 \\
u, c, t \otimes 3 \underline{\text { colours }} & 3 \times 3 \times(+2 / 3) & = & +6 \\
\mathrm{~d}, \mathrm{~s}, \mathrm{~b} \otimes 3 \underline{\text { colours }} & 3 \times 3 \times(-1 / 3) & = & -3 \\
\hline
\end{array}
$$

$\operatorname{SU}(3)_{\text {flavour }}$ or $\operatorname{SU}(6)_{\text {flavour®spin }}$

- Multiplets of comparable mass ... better still, there are patterns in the masses which are well described - see Gell-Mann Okubu formula [11.2]
- Magnetic Moments - see Lecture \#6 [15.4]
- Decay Rates [Close, Lichtenberg]
- Scattering Amplitudes [Close, Lichtenberg]


## The Standard Model

For a Dirac fermion: $\quad L \sim \bar{\psi} \partial \psi=\bar{\psi} \gamma_{\mu} \partial^{\mu} \psi$
We impose local gauge symmetries $\partial^{\mu} \rightarrow \partial^{\mu}-i g X \cdot F^{\mu}$

| "Charge" | Symmetry Group | Generator | Coupling | Bosons |
| :--- | :---: | :---: | :---: | :---: |
| Weak Hypercharge | $U(1)$ | $Y$ | $g_{Y}$ | $B$ |
| Weak Isospin | $S U(2)$ | $T$ | $g_{T}$ | $W^{+}, W^{0}, W^{-}$ |
| Quark Colour | $S U(3)$ | $\lambda$ | $g_{c}$ | $G-8$ gluons |

Left-handed Isospin Doublets:

$$
\begin{aligned}
& L_{L}=\left\{\binom{v_{e}}{e^{-}}_{L},\binom{v_{\mu}}{\mu^{-}}_{L},\binom{v_{\tau}}{\tau^{-}}_{L}\right\} \\
& Q_{L}=\left\{\binom{u}{d}_{L},\binom{c}{s}_{L},\binom{t}{b}_{L}\right\}
\end{aligned}
$$

Right-handed Isospin Singlets:

$$
\begin{array}{ll}
v_{R}=v_{e R}, v_{\mu R}, v_{\tau R} & l_{R}=e_{R}, u_{R}, \tau_{R} \\
u_{R}=u_{R}, c_{R}, t_{R} & d_{R}=d_{R}, s_{R}, b_{R}
\end{array}
$$

- Left-handed refers to the chirality, resulting from the projection operator $\frac{1}{2}\left(1-\gamma_{5}\right)$
- Right-handed refers to the chirality, resulting from the projection operator $\frac{1}{2}\left(1+\gamma_{5}\right)$


## Furthermore

- Any quark is a colour triplet: $q=\left(q_{r}, q_{b}, q_{g}\right)$
- All particle states are Dirac spinors (4 components)

So all of the particles are complicated tensors, and implicitly carry several indices.
Following the observations of neutrino oscillations, the neutrinos must be massive and there must exist right-handed states.
This is claimed to be evidence for New Physics Beyond the Standard Model.
However, the Standard Model (which originally was constructed to include a description of massless neutrinos) can trivially be extended to included right-handed states and masses.

Having said this, because of a hierarchy problem associated with $m(v) \ll m(l, q)$, many theorists like the idea of neutrinos with a Majorana mass term in the Lagrangian (for which the neutrino is its own antiparticle). The masses of the neutrinos are then manifestations of new physics at the GUT or Planck scale.

The conjugates for the doublets are required to transform like row vectors rather than column vectors, and hence look like:

$$
\overline{\mathrm{Q}}_{\mathrm{L}}=\left\{\left(\begin{array}{ll}
\overline{\mathrm{u}} & \overline{\mathrm{~d}}
\end{array}\right)_{L},\left(\begin{array}{ll}
\overline{\mathrm{c}} & \overline{\mathrm{~s}}
\end{array}\right)_{L},(\overline{\mathrm{t}} \quad \overline{\mathrm{~b}})_{L}\right\} \text { rather than }\left\{\binom{\overline{\mathrm{d}}}{-\overline{\mathrm{u}}}_{L},\binom{\overline{\mathrm{~s}}}{-\overline{\mathrm{c}}}_{L},\binom{\overline{\mathrm{~b}}}{-\overline{\mathrm{t}}}_{\mathrm{L}}\right\}
$$

Quantum numbers:

|  | Hypercharge <br> $Y$ | Isospin <br> $T$ | $\lambda$ |
| :--- | :---: | :---: | :--- |
|  |  |  |  |
| $L_{L}$ | -1 | $1 / 2$ | 0 |
| $Q_{L}$ | $1 / 3$ | $1 / 2$ | 1 |
| $V_{R}$ | 0 | 0 | 0 |
| $I_{R}$ | -2 | 0 | 0 |
| $U_{R}$ | $4 / 3$ | 0 | 1 |
| $d_{R}$ | $-2 / 3$ | 0 | 1 |

$Q=T_{3}+1 / 2 Y$

## The Standard Model Lagrangian:

$\mathrm{L} \quad \sim \quad \bar{L}_{L} \gamma_{\mu}\left(\partial^{\mu}-g_{\gamma} B^{\mu} \quad+g_{T} T \cdot W^{\mu}\right)$

|  | $L_{L}$ |
| :--- | :--- |
| $\left.+g_{c} \lambda \cdot G^{u}\right)$ | $Q_{L}$ |
|  | $v_{R}$ |
|  | $I_{R}$ |
| $\left.+g_{c} \lambda \cdot G^{u}\right)$ | $u_{R}$ |
| $\left.+g_{c} \lambda \cdot G^{u}\right)$ | $d_{R}$ |

+ Boson terms
+ Higgs terms, rendered Gauge Invariant, giving Boson mass terms
+ Higgs-Fermion terms, giving Fermion mass terms
(Drop $\mathrm{i}=\sqrt{-1}$; explicitly evaluate Y and ignore $1 / 2$ with $\lambda$ to avoid clutter.)
Baryon and lepton number conservation is explicitly built into the model.
The $v_{\mathrm{R}}$ has no gauge couplings - it is "sterile" - the only interactions it has are with the Higgs.

Could $\mathrm{U}(1)_{\mathrm{Y}}=\mathrm{U}(1)_{\mathrm{Em}}$ ?
No - because terms like $\bar{v}_{L} B^{\mu} v_{L}$ indicate a coupling of the $v$ to the $U(1)_{Y}$ field $B^{\mu}$, and since the $v$ has no electrical charge it does not couple to the exchange boson on $U(1)_{\mathrm{EM}}$, namely the photon.

## Quantum Numbers

In order to be gauge invariant, that is: unaffected by the symmetry transformations corresponding to $\mathrm{U}(1)_{\mathrm{Y}} \otimes \mathrm{SU}(2)_{T} \otimes \mathrm{SU}(3)_{\mathrm{c}}$, terms in the SM Lagrangian must carry no net quantum numbers.

Therefore

- $B$ has $Y=0, T=0$
- $W$ has $Y=0, T=1$ and $Q=+1,0,-1$

For example in a term like $\overline{\mathrm{e}}^{-} \mathrm{W}^{-} v_{\mathrm{e}}$ corresponding to a vertex

$$
\begin{array}{ccccc}
\mathrm{T}_{3}\left(\mathrm{e}^{-}\right) & \rightarrow & \mathrm{T}_{3}\left(\mathrm{v}_{\mathrm{e}}\right) & + & \mathrm{T}_{3}\left(\mathrm{~W}^{-}\right) \\
-1 / 2 & \rightarrow & 1 / 2 & + & -1
\end{array}
$$



## The Higgs Mechanism

The Higgs doublet $\binom{\phi^{+}}{\phi^{0}}$ has $Y=1, T=1 / 2$ (choice ensures $Q=T_{3}+1 / 2 Y$ ).
$\phi^{+}$and $\phi^{0}$ are complex scalars and so represent 4 DoF.
In the Higgs Mechanism, the real part of $\phi^{0}$ is written as $v+h$, where $v$ is the Higgs vev and $h$ is the real Higgs field. So $\phi^{+}$and the imaginary part of $\phi^{0}$ are zero. (Actually, they have zero vev's and the fields correspond to the Goldstone bosons, which are absorbed by the W bosons to provide the longitudinal polarisation).

The Higgs Mechanism results from imposing Gauge Invariance on the Higgs Lagrangian terms:

$$
\phi^{H} \partial_{\mu} \partial^{\mu} \phi+V(\phi)
$$

giving terms like $v^{2} W W$ which look like $W$ mass terms $m(W)^{2} W W$.
By design, the Higgs Mechanism provides masses for the Vector Bosons $\left\{\mathrm{W}^{+}, \mathrm{W}^{0}, \mathrm{~W}^{-}\right\}$while causing the mixing of the $W^{0}$ and $B$ gauge fields resulting in the $Z$ and $\gamma$ fields.

## Fermion Masses

The obvious expression for a Langrangian representing fermion mass has the form $m \bar{\psi} \psi$. If we relate this to the particles of the SM , we must consider $\mathrm{e}_{\mathrm{L}}$ and $\mathrm{e}_{\mathrm{R}}$ as being distinct. If we take the Dirac operator, $\psi$, to be $e_{\mathrm{L}}=\frac{1}{2}\left(1-\gamma_{5}\right)$ e, then the chirality operator, when it acts on the conjugate, $\bar{\psi}$, it reverses the chirality and requires the right-handed electron operator. So a mass term for the electron would look like $m \bar{e}_{R} e_{L} \ldots$ and this does not conserve gauge invariance.

We recall for $\operatorname{SU}(2)$, that the $\overline{\mathbf{2}}$ transforms like the $\mathbf{2}$.
This enables us to construct a gauge invariant part of the Lagrangian describing the fermion masses (ignoring the different couplings which lead to different fermion masses):

$$
\overline{\mathrm{L}}_{\mathrm{L}}\binom{\phi^{+}}{\phi^{0}} \mathrm{I}_{\mathrm{R}}+\overline{\mathrm{L}}_{\mathrm{L}}\binom{\bar{\phi}^{0}}{-\phi^{-}} \mathrm{v}_{\mathrm{R}}+\overline{\mathrm{Q}}_{\mathrm{L}}\binom{\phi^{+}}{\phi^{0}} \mathrm{~d}_{\mathrm{R}}+\overline{\mathrm{Q}}_{\mathrm{L}}\binom{\bar{\phi}^{0}}{-\phi^{-}} \mathrm{u}_{\mathrm{R}}+\text { hc + non-diagonal terms }
$$

All the terms above give rise to zero net quantum numbers:
For example, consider the first term $\sim \bar{v}_{L} \phi^{+} I_{R}+\bar{I}_{L} \phi^{0} I_{R}$
The above expression needs to be repeated for each of the generations $\left(v_{e}, e\right),\left(v_{\mu}, \mu\right),\left(v_{\tau}, \tau\right),(u, d)$, (c,s), (t,b).
"Non-diagonal" terms which combine different generations give rise to mixing, as found in the CKM matrix.

The second term is new for the Standard Model, corresponding to Dirac masses for the neutrinos (if this is the correct mechanism for neutrino masses).

The fermion mass terms appear when the $\phi$ fields are replaced by their vev's, giving terms like $v \bar{l}_{L} l_{R}$ which looks like the lepton mass term $m(l) \bar{l}_{\mathrm{L}} \mathrm{I}_{\mathrm{R}}$.

The different assignments of weak quantum numbers to the left- and right-handed fermions allows the possibility of $\mathrm{C}, \mathrm{P}$ and CP violation.
As CPT is conserved in Quantum Field Theories, this implies T can also be violated.

## Beyond the Standard Model

Undesirable features of the Standard Model include:

- Large number of unconstrained constants (couplings, charges)
- No explanation of generations
- No explanation as to charge quantisation: $Q(u)=-2 / 3 Q\left(e^{-}\right)$and $Q(d)=1 / 3 Q\left(e^{-}\right)$

It would seem elegant to contain all of the SM in some single theory (group) - rather than as the product of three seemingly disconnected groups.
While the Electroweak force is supposedly unified, it nevertheless represents the product of two groups $\mathrm{SU}(2)_{\mathrm{T}}$ and $\mathrm{U}(1)_{\mathrm{Y}}$, albeit that they are mixed through the Higgs Mechanism.

If the coupling constants of the three groups $\mathrm{U}(1)_{\mathrm{Y}}, \mathrm{SU}(2)_{\mathrm{T}}$ and $\mathrm{SU}(3)_{c}$ are evolved via the Renormalisation Group Equations (RGE), they meet at a scale of $\sim 10^{15} \mathrm{GeV}$.
It turns out that this convergence is even more precise if SuperSymmetry is included.

## SU(5) [18]

The fundamental representation is taken as $5=\left(\begin{array}{l}d_{r} \\ d_{b} \\ d_{g} \\ e^{+} \\ \bar{v}_{e}\end{array}\right)_{R}$
The particles of the Standard Model (in the absence of neutrino mass and hence $v_{\mathrm{R}}$ ) can be contained in multiplets of $\operatorname{SU}(5)$ :

$$
\overline{\mathbf{5}}=\left(\begin{array}{c}
\bar{d}_{r} \\
\bar{d}_{\mathrm{d}_{\mathrm{b}}} \\
\bar{d}_{g} \\
e^{-} \\
-v_{e}
\end{array}\right)_{\mathrm{L}} \quad \mathbf{1 0}=\left(\begin{array}{ccccc}
0 & \bar{u}_{g} & -\bar{u}_{\mathrm{b}} & -u_{r} & -d_{r} \\
-\bar{u}_{g} & 0 & \bar{u}_{\mathrm{r}} & -u_{\mathrm{b}} & -d_{\mathrm{b}} \\
\bar{u}_{\mathrm{b}} & -\bar{u}_{r} & 0 & -u_{g} & -d_{g} \\
\mathrm{u}_{\mathrm{r}} & u_{\mathrm{b}} & u_{g} & 0 & -e^{+} \\
\mathrm{d}_{\mathrm{r}} & d_{\mathrm{b}} & d_{g} & e^{+} & 0
\end{array}\right)_{\mathrm{L}} \text { Reproduced for each of the three generations. }
$$

Where the $\mathbf{1 0}$ an antisymmetric multiplet derived from 5 $\mathbf{5}$ :$\otimes$

$\square$
$5 \otimes 5=$ 15
$\oplus$
$\oplus$
10
Totally
Antisymmetric

There are $5^{2}-1=24$ gauge bosons. In terms of the fundamental representation they have the form:

|  | $d_{r}$ | $d_{b}$ | $d_{g}$ | $e^{+}$ | $\bar{v}_{e}$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $d_{r}$ | $g, \gamma, Z$ | $g$ | $g$ | $\bar{X}_{r}$ | $\bar{Y}_{r}$ |
| $d_{b}$ | $g$ | $g, \gamma, Z$ | $g$ | $\bar{X}_{b}$ | $\bar{Y}_{b}$ |
| $d_{g}$ | $g$ | $g$ | $g, \gamma, Z$ | $\bar{X}_{g}$ | $\bar{Y}_{g}$ |
| $\mathrm{e}^{+}$ | $X_{r}$ | $X_{b}$ | $X_{g}$ | $\gamma, Z$ | $W^{+}$ |
| $\bar{v}_{e}$ | $Y_{r}$ | $Y_{b}$ | $Y_{g}$ | $W^{-}$ | $Z$ |

So there are 8 gluons, 3 of $\mathrm{W}^{+}, \mathrm{W}^{-}, \mathrm{Z}$ and $1 \gamma$, along with $12 \mathrm{X}, \mathrm{Y}$ bosons.
The $\mathrm{X}, \mathrm{Y}$ bosons are coloured and transform leptons to quarks and v.v. - they are leptoquarks.
As we saw in Lecture \#2, the $X, Y$ bosons can mediate proton decay:

$B$ and $L$ are not conserved, but $B-L$ is.

The good points about $\operatorname{SU}(5)$ are:

- It is the smallest group which contains $U(1)^{r} \otimes S U(2)_{T} \otimes S U(3)_{c}$, and therefore has the greatest predictivity.
- $Q=T_{3}+1 / 2 Y$ and $T_{3}$ and $Y$ are now both generators of $S U(5)$ and hence traceless.
(In the Standard Model, Y was a generator of $\mathrm{U}(1)_{\mathrm{Y}}$, and was not traceless).
Hence $Q$ is traceless and in the fundamental representation this leads:

$$
Q\left(d_{r}\right)+Q\left(d_{\mathrm{b}}\right)+Q\left(d_{\mathrm{g}}\right)+Q\left(\mathrm{e}^{+}\right)+\mathrm{Q}\left(\bar{v}_{\mathrm{e}}\right)=0 \Rightarrow 3 Q(\mathrm{~d})+Q\left(\mathrm{e}^{+}\right)=0 \Rightarrow Q(\mathrm{~d})=-1 / 3 Q\left(\mathrm{e}^{+}\right) \text {etc }
$$

- It predicts $\sin ^{2} \theta_{w}$ very accurately.
- The couplings of the low-energy gauge group representations converge at a scale ~M(X).
- Since there is only room for one neutrino type in the 5 and 10, i.e. there is no $v_{R}$, hence the only way to generate a mass term is via a Majorana term which requires $v=\bar{v}$ - which violates $\mathrm{B}-\mathrm{L}$ conservation. Hence the neutrino must be massless. But ...

The problems are:

- The neutrino appears to have mass.
- The predicted decay rate for the proton is much larger than the current limits.
- There is no explanation of the three generations.

So despite its elegance, $\operatorname{SU}(5)$ is now excluded.

## Other Symmetry Groups [24, 27]

Another possibility is $\mathrm{SO}(10)$ which contains $\mathrm{SU}(5)$ - this is not yet excluded.
Going further, other possibilities include the Exceptional Groups: $\mathrm{E}_{6}$ and $\mathrm{E}_{8}$.
The latter is of relevance to the heterotic superstring (combining bosonic and fermionic modes).
These groups are associated with the algebra of Octonians, which are generalisations of $\mathbf{i}=\sqrt{-1}$.
$\mathrm{E}_{6}$ contains $\mathrm{SO}(10)$.


## SuperSymmetry

SUSY provides an extension of the Poincaré groups (associated with Translations, Rotations and Lorentz Boosts).
It is the one remaining symmetry in QFT, not exploited in the Standard Model Lagrangian, whereby transformations take place which treat bosons and fermions as indistinguishable states.

Crudely speaking, the generator can be thought of as the square root of the momentum operator!
The corresponding super-fields multiplets are:

$$
\text { gauge: }\binom{J=1}{J=\frac{1}{2}} \text { chiral (matter): }\binom{J=\frac{1}{2}}{J=0}
$$

In many SUSY models, there is a conserved quantity, R-parity, which distinguishes between the standard particles and their super-partners.
This ensures that SUSY particles can only be created in pairs, and that once created, a SUSY particle will always leave a SUSY particle in the final state, leading to a stable Lightest Supersymmetric Particle (LSP), commonly considered to be the neutralino - a mixture of the photino, Zino and neutral Higgsinos.
These may be one of the constituents of Dark Matter.

