Lecture Notes on Neutrino oscillations in matter

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In this note we discuss the phenomenology of neutrino oscillations in matter. We introduce the MSW mechanism in the framework of oscillations between two families, where the formalism is simple but still adequate to illustrate many important consequences and to interpret the experimental results for solar neutrinos.

We start by recalling that in the case of vacuum oscillation, the time evolution of flavour eigenstates satisfies the Schroedinger equation [1]

$$i\frac{d}{dt}\begin{pmatrix} \nu_{\rm e} \\ \nu_\mu \end{pmatrix} = H_V \begin{pmatrix} \nu_{\rm e} \\ \nu_\mu \end{pmatrix} \tag{1}$$

where

$$H_V = \left(\frac{\Delta m^2}{4E}\right)\begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \tag{2}$$

When neutrinos travel through a dense medium (e.g., in the Sun or in the Earth), their propagation can be significantly modified by the coherent forward scattering from particles they encounter along the way. As a result, the oscillation probability can be rather different than it is in vacuum. The flavour-changing mechanism in matter was named after Mikhaev, Smirnov and Wolfenstein (MSW), who first pointed out [1] that there is an interplay between flavour-non-changing neutrino-matter interactions and neutrino mass and mixing. The MSW effect stems from the fact that electron

[1] Eq.1 can be easily obtained from the time-evolution of the mass-eigenstates, which in matrix form can be written as

$$i\frac{d}{dt}\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \simeq \begin{pmatrix} m_1^2/2p & 0 \\ 0 & m_2^2/2p \end{pmatrix}\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} + \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix},$$

after substitution of ν_1 and ν_2 with their expression in terms of $\nu_{\rm e}$ and ν_μ, i.e.,

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}.$$

Note that multiple of the identity can be omitted as they introduce a constant phase factor which does not affect oscillations.

neutrinos (and antineutrinos) have different interactions with matter compared to other neutrinos flavours. In particular, ν_{e} can have both charged current and neutral current elastic scattering with electrons, while ν_μ or ν_τ have only neutral current interactions with electrons. This fact gives rise to an extra-potential $\mathrm{V_e} = \pm\sqrt{2}\mathrm{G_F N_e}$ [2], where N_e is the electron density in matter, G_F is the Fermi constant, and the positive(negative) sign applies to electron-neutrino(antineutrinos).

Therefore, the effective Hamiltonian which governs the propagation of neutrinos in matter, H_M, contains an extra ν_{e}-ν_{e} element, and can be written as

$$H_M = \left(\frac{\Delta m^2}{4E}\right)\begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} + \begin{pmatrix} \mathrm{V_e} & 0 \\ 0 & 0 \end{pmatrix} \tag{3}$$

Without modifying the physics, we can subtract the following multiple of the identity from Eq. 3

$$\begin{pmatrix} \mathrm{V_e}/2 & 0 \\ 0 & \mathrm{V_e}/2 \end{pmatrix}$$

to obtain

$$H_M = \left(\frac{\Delta m^2}{4E}\right)\begin{pmatrix} -\cos 2\theta + \mathrm{A} & \sin 2\theta \\ \sin 2\theta & \cos 2\theta - \mathrm{A} \end{pmatrix} \tag{4}$$

with

$$A = \pm\frac{2\sqrt{2}G_F N_e E}{\Delta m^2}.$$

The solution of the corresponding Schroedinger equation is simple in the case where the matter density is constant. In this case, we can simply rediagonalise H_M to obtain the mixing matrix and mass eigenstates in matter via a rotation matrix, similar to that for vacuum. If we note the effective mixing angle in matter as θ_m and the effective difference of squared masses as Δm^2_m, we can write the Hamiltonian in matter using the same form as the vacuum Hamiltonian

$$H_M = \left(\frac{\Delta m^2_m}{4E}\right)\begin{pmatrix} -\cos 2\theta_{\mathrm{m}} & \sin 2\theta_{\mathrm{m}} \\ \sin 2\theta_{\mathrm{m}} & \cos 2\theta_{\mathrm{m}} \end{pmatrix} \tag{5}$$

which leads to the usual functional dependence of the oscillation probability

$$P(\nu_{\mathrm{e}} \to \nu_\mu) = sin^2 2\theta_m sin^2\left(\frac{\Delta m^2_m L}{4E}\right).$$

By equating Eq. 4 and Eq. 5, we can derive the expression for the effective mixing parameters in matter

$$\Delta m_m^2 = C\Delta m^2,$$

$$sin2\theta_m = \frac{sin2\theta}{C},$$

where

$$C = \sqrt{(cos2\theta - A)^2 + sin^2 2\theta}.$$

These formula are useful to understand the following important consequences of the MSW effect:

1. long baselines or high matter densities are required to observe significant matter effects (in the limit $\Delta m_m^2 L/(4E) << 1$ the vacuum probabilities can be retrieved);

2. under the resonant condition $cos2\theta = A$, oscillations can be significantly enhanced, irrespectively of the value of θ, therefore even if the vacuum oscillation probability is very small. Since $A = L_V/L_e$, where $L_V = 4\pi E/\Delta m^2$ is the vacuum oscillation length and $L_e = 4\pi/(2\sqrt{2}G_F N_e)$ is the electron-neutrino interaction length, the resonance condition is

   $$L_V = L_e cos2\theta.$$

   For example, as the density in the core of the Sun is $\rho(\text{core}) \sim 100 gcm^{-3}$ or $N_e(core) \sim 3 \times 10^{31} m^{-3}$, $L_e(core)$ is about $3 \times 10^5 m$, much smaller than the solar radius, which is about $7 \times 10^8 m$.[2] So solar neutrinos with energy exceeding

   $$E_{min} = \Delta m^2 \frac{cos2\theta}{2\sqrt{2}G_F N_e(core)}$$

   will always pass through the resonance region. For $\theta_{sol} \sim 30°$ and $\Delta m_{sol}^2 \sim 8 \times 10^{-5} eV^2$ (LMA-MSW solution to the solar neutrino problem) E_{min} is about $1MeV$.

[2]The solar-density decreases roughly exponentially with the distance from the center, so we have to deal with varying matter densities. A way to approach the problem, which is used in numerical calculations, is to solve the Schroedinger equation for layers of constant density and then patch the solutions together.

3. oscillation probabilities for neutrino and antineutrinos can be different due to matter effects (because of the $\pm$ sign in front of A), even if neutrino interactions with matter do not violate CP, even if the mixing matrix is real;

4. the resonant condition occurs if $A > 0$, which in turns depend on the sign of Δm^2. This dependence on the sign of Δm^2 can be used to determine the neutrino mass hierarchy. For example, long baseline experiments at accelerators, which look for $\nu_\mu \to \nu_{\rm e}$ oscillations in the region of $\Delta m^2 \sim \Delta m^2_{23}$, are sensitive to the mass hierarchy through matter effects in the Earth if the baseline is sufficiently long and the energy is sufficiently high, so that matter effects are significant. Baselines of the order of 1000 Km are required for typical accelerator neutrino energies, O(GeV).

References

[1] L. Wolfenstein, Phys. Rev. D17 (1978) 2369; S.P. Mikhaev and A.Y. Smirnov, Sov.J.Nucl.Phys. 42 (1985) 913; S.P. Mikhaev and A.Y. Smirnov, Nuovo Cimento C9 (1986) 17.

[2] For a formal derivation, see for example: C. Giunti and C.W. Kim, "Fundamentals of Neutrino Physics and Astrophysics", Oxford University Press, 2007.